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A NOTE ON THE VANISHING OF $H^n(G, \mathbb{Z}G)$

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If the group G is of type $F(n)$ [$\equiv FP_n$ and finitely presented] necessary and sufficient conditions are given for $H^n(G, \mathbb{Z}G)=0$. These sharpen a result in a previous paper of ours. In particular, if G is finitely presented and simply connected at ∞ at each end, then $H^2(G, \mathbb{Z}G)=0$.

A group G is *of type* $F(n)$ if there exists a $K(G,1)$ CW complex X having finite n-skeleton. Our paper [2] deals with the relationship between $H^k(G, \mathbb{Z}G)$ and the homology inverse sequence of the end of the locally finite universal cover $\tilde{X}^n$, where $k \le n$. The results in [2] are sharp when $k<n$. It turns out that in several applications (explained at the end of this note) a sharp result is useful in case $k=n$. This note contains the details: it should be regarded as an addendum to [2]. For reference we state the main theorem and corollary of [2] in sharpened form (the changes are in (iii) and (iv)):

Theorem. *Let G be a group of type $F(n)$, and let X be a $K(G,1)$ CW complex having finite n-skeleton.*

(i) *For $k \le n$, $H^k(G, \mathbb{Z}G)$ mod torsion is free abelian if and only if $H_{k-1}(\varepsilon\tilde{X}^n)$ is semi-stable ($\equiv$ Mittag-Leffler).*

(ii) *For $k \le n$, $H^k(G, \mathbb{Z}G)$ is torsion free if and only if $H_{k-2}(\varepsilon\tilde{X}^n)$ is pro-torsion free.*

(iii) *For G infinite, and $k \le n$, $H^k(G, \mathbb{Z}G)$ is a torsion group if and only if $\bar{H}_{k-1}(\varepsilon\tilde{X}^n)$ is pro-finite.*

(iv) *For G infinite, and $k \le n$, $H^k(G, \mathbb{Z}G)$ mod torsion is free abelian of finite rank ϱ if and only if $\bar{H}_{k-1}(\varepsilon\tilde{X}^n)$ mod torsion is stable with free abelian inverse limit of finite rank ϱ.*

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Corollary. *With G as in the Theorem:*

(i) *$H^r(G,\mathbb{Z}G)=0$ for all $r<n$ and $H^n(G,\mathbb{Z}G)$ is free abelian if and only if $\bar{H}_r(\varepsilon\tilde{X}^n)$ is pro-trivial for all $r\leq n-2$ and $H_{n-1}(\varepsilon\tilde{X}^n)$ is semi-stable.*

(ii) *$H^r(G,\mathbb{Z}G)=0$ for all $r<n$ and $H^n(G,\mathbb{Z}G)$ is torsion free if and only if $\bar{H}_r(\varepsilon\tilde{X}^n)$ is pro-trivial for all $r\leq n-2$.*

(iii) *$H^r(G,\mathbb{Z}G)=0$ for all $r\leq n$ if and only if $\bar{H}_r(\varepsilon\tilde{X}^n)$ is pro-trivial for all $r\leq n-2$ and $\bar{H}_{n-1}(\varepsilon\tilde{X}^n)$ is pro-finite.*

(iv) *$H^r(G,\mathbb{Z}G)=0$ for all $r\leq n-1$ and $H^n(G,\mathbb{Z}G)$ is free abelian of finite rank ϱ if and only if $\bar{H}_r(\varepsilon\tilde{X}^n)$ is pro-trivial for all $r\leq n-2$ and $H_{n-1}(\varepsilon\tilde{X}^n)$ mod torsion is stable with free abelian inverse limit of finite rank ϱ.*

Notation. $\bar{H}_*$ is reduced homology. $\varepsilon\tilde{X}^n$ denotes the inverse system $\{\tilde{X}^n\setminus K_\alpha\}$ where $\tilde{X}^n$ is the (locally compact) n-skeleton of the universal cover $\tilde{X}$ of X and K_α ranges over the finite subcomplexes of $\tilde{X}$. It is enough to pick an exhausting subsequence of $\{K_\alpha\}$. For the terminology of pro-abelian groups, see [2].

Proofs. The only change from [2] is the replacement of $<$ by $\leq$ throughout (iii) and (iv). This is justified by using 3.7, 3.8 and 4.1 of [2], together with the following Lemma. □

Lemma. *With G and n as above, $H^n(G,\mathbb{Z}G)$ is isomorphic to* $\ker(H^n_c(\tilde{X}^n)\xrightarrow{\alpha_*} H^n(\tilde{X}^n))$ *where α_* is the natural homomorphism from cohomology with compact supports to ordinary cohomology.*

Proof of Lemma. We have a commutative diagram of cochain complexes and homomorphisms of graded groups:

$$\begin{tikzcd}
\mathscr{C}_{\mathrm{c}} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathscr{C}(n) \arrow[d, "\sigma"] \\
\mathscr{C}(G) \arrow[r, "\gamma"] & \mathscr{C}(n+1)
\end{tikzcd}$$

The elements of this diagram are displayed in Diagram 1.

α is inclusion. β and σ are the obvious isomorphisms, except in dimension $n+1$ where they are zero. $\gamma:\mathrm{Hom}_G(C_k(\tilde{X}^{n+1}),\mathbb{Z}G)\to\mathrm{Hom}(C_k(\tilde{X}^{n+1}),\mathbb{Z})$ is the usual monomorphism (see [1; p. 209, line 6]). α and γ are morphisms of cochain complexes, but β and σ are not, since the leftmost squares involving β and σ (in the diagram) do not commute.

$$\begin{array}{lccccc}
\mathscr{C}_c: & 0 & \longleftarrow & \mathrm{Hom}_c(C_n(\tilde X^n),\mathbb{Z}) & \xleftarrow{\delta} & \mathrm{Hom}_c(C_{n-1}(\tilde X^n),\mathbb{Z}) \leftarrow\cdots \\
 & \downarrow\alpha & & \downarrow\alpha & & \downarrow\alpha \\
\mathscr{C}(n): & 0 & \longleftarrow & \mathrm{Hom}(C_n(\tilde X^n),\mathbb{Z}) & \xleftarrow{\delta} & \mathrm{Hom}(C_{n-1}(\tilde X^n),\mathbb{Z}) \leftarrow\cdots \\
 & \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma \\
\mathscr{C}(n+1): & \mathrm{Hom}(C_{n+1}(\tilde X^{n+1}),\mathbb{Z}) & \xleftarrow{\delta} & \mathrm{Hom}(C_n(\tilde X^{n+1}),\mathbb{Z}) & \xleftarrow{\delta} & \mathrm{Hom}(C_{n-1}(\tilde X^{n+1}),\mathbb{Z}) \leftarrow\cdots \\
 & \uparrow\gamma & & \uparrow\gamma & & \uparrow\gamma \\
\mathscr{C}(G): & \mathrm{Hom}_G(C_{n+1}(\tilde X^{n+1}),\mathbb{Z}G) & \xleftarrow{\delta} & \mathrm{Hom}_G(C_n(\tilde X^{n+1}),\mathbb{Z}G) & \xleftarrow{\delta} & \mathrm{Hom}_G(C_{n-1}(\tilde X^{n+1}),\mathbb{Z}G) \leftarrow\cdots \\
 & \uparrow\beta & & \uparrow\beta & & \uparrow\beta \\
\mathscr{C}_c: & 0 & \longleftarrow & \mathrm{Hom}_c(C_n(\tilde X^n),\mathbb{Z}) & \xleftarrow{\delta} & \mathrm{Hom}_c(C_{n-1}(\tilde X^n),\mathbb{Z}) \leftarrow\cdots
\end{array}$$

Diagram 1.

Now let $f\in\mathrm{Hom}_c(C_n(\tilde X^n),\mathbb{Z})$ represent $[f]\in\ker\alpha_*$. Then $\alpha(f)$ is a coboundary, hence also $\sigma\alpha(f)\equiv\gamma\beta(f)$. So $\gamma\beta(f)$ is a cocycle, hence also $\beta(f)$. Moreover, if f is a coboundary, so is $\beta(f)$. Thus β induces $\beta_*:\ker\alpha_*\to H^n(G,\mathbb{Z}G)$. Clearly, β_* is mono. To see that β_* is epi, let $\bar f\in\mathrm{Hom}_G(C_n(\tilde X^{n+1}),\mathbb{Z}G)$ be a cocycle representing $[\bar f]\in H^n(G,\mathbb{Z}G)$. Then $\gamma(\bar f)$ is a cocycle, hence a coboundary, since $H^n(\tilde X^{n+1})=0$. β is an isomorphism, so $\gamma(\bar f)=\gamma\beta(f)$ for some $f\in\mathrm{Hom}_c(C_n(\tilde X^n),\mathbb{Z})$. So $\sigma\alpha(f)\equiv\gamma\beta(f)\equiv\gamma(\bar f)$ is a coboundary, hence also $\alpha(f)$. So $[f]\in\ker\alpha_*$. $\beta_*[f]=[\beta(f)]=[\bar f]$. $\square$

Application. G is finitely presented. Suppose one wishes to prove that $H^2(G,\mathbb{Z}G)=0$ (see, for example [3] or [4]). If the usual spectral sequence methods are inapplicable, one can often prove topologically that G is simply connected at ∞ at each end: i.e., for a finite 2-complex X^2 whose fundamental group is G, the inverse sequence $\pi_1(\varepsilon\tilde X^2,r)$ is pro-trivial for each base ray r. See for example [5], [7], [8] or [9]. It then follows that $H_1(\varepsilon\tilde X^2)$ is pro-trivial, hence, by (iii) and (iv) of the Theorem, that $H^2(G,\mathbb{Z}G)=0$. This can only be obtained from [2] when G is of type $F(3)$ (rather than $F(2)\equiv$ finitely presented). For the case of G one-ended, this particular application can also be drawn from [5] as corrected in [6]; however, the method used there does not obviously generalize to the $F(n)$ case.

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